

# Blow-up, zero $\alpha$ limit and the Liouville type theorem for the Euler-Poincaré equations

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## Abstract

In this paper we study the Euler-Poincaré equations in  $\mathbb{R}^N$ . We prove local existence of weak solutions in  $W^{2,p}(\mathbb{R}^N)$ ,  $p > N$ , and local existence of unique classical solutions in  $H^k(\mathbb{R}^N)$ ,  $k > N/2 + 3$ , as well as a blow-up criterion. For the zero dispersion equation ( $\alpha = 0$ ) we prove a finite time blow-up of the classical solution. We also prove that as the dispersion parameter vanishes, the weak solution converges to a solution of the zero dispersion equation with sharp rate as  $\alpha \rightarrow 0$ , provided that the limiting solution belongs to  $C([0, T]; H^k(\mathbb{R}^N))$  with  $k > N/2 + 3$ . For the *stationary weak solutions* of the Euler-Poincaré equations we prove a Liouville type theorem. Namely, for  $\alpha > 0$  any weak solution  $\mathbf{u} \in H^1(\mathbb{R}^N)$  is  $\mathbf{u} = 0$ ; for  $\alpha = 0$  any weak solution  $\mathbf{u} \in L^2(\mathbb{R}^N)$  is  $\mathbf{u} = 0$ .

**Key words:** finite time blow-up, zero dispersion limit, Liouville type theorem, Euler-Poincaré equations, Camassa-Holm equation

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## 1 Introduction

We consider the following Euler-Poincaré equations in  $\mathbb{R}^N$ :

$$(EP) \begin{cases} \partial_t \mathbf{m} + (\mathbf{u} \cdot \nabla) \mathbf{m} + (\nabla \mathbf{u})^\top \mathbf{m} + (\operatorname{div} \mathbf{u}) \mathbf{m} = 0, \\ \mathbf{m} = (1 - \alpha \Delta) \mathbf{u}, \\ \mathbf{u}_0(x) = \mathbf{u}_0, \end{cases}$$

where  $\mathbf{u} : \mathbb{R}^N \rightarrow \mathbb{R}^N$  is the velocity,  $\mathbf{m} : \mathbb{R}^N \rightarrow \mathbb{R}^N$  represents the momentum, constant  $\sqrt{\alpha}$  is a length scale parameter,  $(\nabla \mathbf{u})^\top =$  the transpose of  $(\nabla \mathbf{u})$ . The Euler-Poincaré equations arise in diverse scientific applications and enjoy several remarkable properties both in the one-dimensional and multi-dimensional cases.

The Euler-Poincaré equations were first studied by Holm, Marsden, and Ratiu in 1998 as a framework for modeling and analyzing fluid dynamics [18, 19], particularly for nonlinear shallow water waves, geophysical fluids and turbulence modeling. There are intensive researches on analogs viscous or inviscid, incompressible Lagrangian averaged models. We refer to [7, 12, 26] for results on Navier-Stokes- $\alpha$  model in terms of existence and uniqueness, zero  $\alpha$  limit to the Navier-Stokes equations, global attractor, etc. We refer to [2, 20, 23] for results on analysis and simulation of vortex sheets with Birkhoff-Rott- $\alpha$  or Euler- $\alpha$  approximation.

For one-dimension, the Euler-Poincaré equations coincide with the dispersion-less case of Camassa-Holm (CH) equation [4]:

$$(CH) \quad \partial_t m + 2u \partial_x m + \partial_x u m = 0, \quad m = (1 - \alpha \partial_{xx})u.$$

The solutions to (CH) are characterized by a discontinuity in the first derivative at their peaks and are thus referred to as peakon solutions. (CH) is completely integrable with a bi-Hamiltonian structure and their peakon solutions are true solitary waves that emerge from the initial data. Peakons exhibit a remarkable stability—their identity is preserved through nonlinear interactions, see, e.g. [4, 22]. There are many comprehensive analysis on (CH) in the literature. We refer to a review paper [25] for a survey of recent results on well-posedness and existence of local and global weak solutions for (CH). The existence of a global weak solution and uniqueness was proven in [3, 6, 10, 8, 29]. A class of the so called weak-weak solution was studied in [29]. The breakdown of the solution for (CH) was studied in [24].

The Euler-Poincaré equations have many further interpretations beyond fluid applications. For instance, in 2-D, it is exactly the same as the averaged template matching equation for computer vision (see, e.g., [14, 17, 21]). The Euler-Poincaré equations also has important applications in computational anatomy (see, e.g. [22, 30]). The Euler-Poincaré equations can also be regarded as an evolutionary equation for a geodesic motion on a diffeomorphism group and it is associated with Euler-Poincaré reduction via symmetry [1, 11, 22, 15, 30]. We refer to a recent book [22] for a comprehensive review on the subject.

The organization of the paper is as follows. In Section 2, we give some preliminary discussions of the Euler-Poincaré equations and we state a theorem on local existence of weak solution in  $W^{2,p}(\mathbb{R}^N)$ ,  $p > N$ , and local existence of unique classical solutions in  $H^k(\mathbb{R}^N)$ ,  $k > N/2 + 3$ .

In Section 3, we prove a theorem on a blow-up criterion, as well as, a theorem on finite time blow-up of the classical solution for the zero dispersion equation. For classic solutions with reflection symmetry, the divergence  $\nabla \cdot \mathbf{u}$  satisfy a Riccati equation at the invariant point under the reflection transformation and hence there is a finite time blow up if the divergence is initially negative.

In Section 4, we prove that as the dispersion parameter  $\alpha$  vanishes, the weak solution converges to a solution of the zero dispersion equation with a sharp rate as  $\alpha \rightarrow 0$ , provided that the limiting solution belongs to  $C([0, T]; H^k(\mathbb{R}^N))$  with  $k > N/2 + 3$ .

Finally, for the *stationary weak solutions* of the Euler-Poincaré equations we prove a Liouville type theorem in Section 5. For  $\alpha > 0$ , we prove that any weak solution  $\mathbf{u} \in H^1(\mathbb{R}^N)$  is  $\mathbf{u} = 0$ . For  $\alpha = 0$ , any weak solution  $\mathbf{u} \in L^2(\mathbb{R}^N)$  is  $\mathbf{u} = 0$ . This is a surprising result, as all the previous Liouville type results are for dissipative systems. This is the first Liouville type theorem for non-dissipative systems.

We also give a proof of the local existence and uniqueness theorem in Appendix.

## 2 Preliminaries and local existence

In this section, we discuss some mathematical structures of (EP) and then we state a local existence theorem for the weak solution and the classic solution. We refer to [17, 22] for more in-depth discussions on (EP).

(EP) can be recast as

$$\partial_t \mathbf{m} + \nabla \cdot (\mathbf{u} \otimes \mathbf{m}) + (\nabla \mathbf{u})^\top \mathbf{m} = 0. \quad (1)$$

The last term above can be written in a conservative/tensor form

$$\begin{aligned} \sum_{j=1}^N \partial_i u_j m_j &= \sum_{j=1}^N \partial_i u_j u_j - \alpha \sum_{j,k=1}^N \partial_i u_j \partial_k^2 u_j \\ &= \frac{1}{2} \partial_i |\mathbf{u}|^2 - \alpha \sum_{j,k=1}^N \partial_k (\partial_i u_j \partial_k u_j) + \alpha \sum_{j,k=1}^N \partial_k \partial_i u_j \partial_k u_j \\ &= \frac{1}{2} \partial_i |\mathbf{u}|^2 - \alpha \sum_{j,k=1}^N \partial_j (\partial_i u_k \partial_j u_k) + \frac{\alpha}{2} \sum_{j,k=1}^N \partial_i (\partial_k u_j)^2 \\ &= \sum_{j=1}^N \partial_j \left( \frac{1}{2} \delta_{ij} |\mathbf{u}|^2 - \alpha \partial_i \mathbf{u} \cdot \partial_j \mathbf{u} + \frac{\alpha}{2} \delta_{ij} |\nabla \mathbf{u}|^2 \right). \end{aligned}$$

Set stress-tensor

$$T_{ij} = m_i u_j + \frac{\delta_{ij}}{2} |\mathbf{u}|^2 - \alpha \partial_i \mathbf{u} \cdot \partial_j \mathbf{u} + \frac{\alpha \delta_{ij}}{2} |\nabla \mathbf{u}|^2.$$

Then (EP) becomes

$$\partial_t m_i + \sum_{j=1}^N \partial_j T_{ij} = 0. \quad (2)$$

The first term in  $T_{ij}$  involves a second order derivative of  $\mathbf{u}$  and it can be rewritten as

$$m_i u_j = u_i u_j + \alpha \sum_{k=1}^N \partial_k u_i \partial_k u_j - \alpha \sum_{k=1}^N \partial_k (u_j \partial_k u_i).$$

The symmetric part of tensor  $T$  is given by

$$T^a = \mathbf{u} \otimes \mathbf{u} + \alpha \nabla \mathbf{u} \nabla \mathbf{u}^\top - \alpha \nabla \mathbf{u}^\top \nabla \mathbf{u} + \frac{1}{2}(|\mathbf{u}|^2 + \alpha |\nabla \mathbf{u}|^2) \text{Id} \quad (3)$$

and the remainder terms in  $T$  are given by

$$T_{i,j}^b = -\alpha \sum_{k=1}^N \partial_k (u_j \partial_k u_i). \quad (4)$$

Hence  $T = T^a + T^b$ . In view of this, the natural definition of the weak solution of (EP) would be:

**Definition 1**  $\mathbf{u} \in L^\infty(0, T; H_{loc}^1(\mathbb{R}^N))$  is a weak solution of (EP) with initial data  $\mathbf{u}_0 \in H_{loc}^1(\mathbb{R}^N)$  if the following equation holds for all vector field  $\phi(x, t)$  such that  $\phi(\cdot, t) \in C_0^\infty(\mathbb{R}^N)$  for all  $t \in [0, T)$  and  $\phi(x, \cdot) \in C_0^1([0, T))$  for all  $x \in \mathbb{R}^N$

$$\begin{aligned} & \int_0^T \int_{\mathbb{R}^N} (\mathbf{u} \cdot \phi_t + \alpha \nabla \mathbf{u} : \nabla \phi_t) dx dt + \int_{\mathbb{R}^N} (\mathbf{u}_0 \cdot \phi(\cdot, 0) + \alpha \nabla \mathbf{u}_0 : \nabla \phi(\cdot, 0)) dx \\ & + \int_0^T \int_{\mathbb{R}^N} T^a : \nabla \phi(x, t) dx dt + \alpha \sum_{i,j,k=1}^N \int_0^T \int_{\mathbb{R}^N} u_j \partial_k u_i \partial_j \partial_k \phi_i dx dt = 0, \end{aligned} \quad (5)$$

where  $T^a$  is given by (3).

(EP) also has a natural Hamiltonian structure. Set

$$\mathcal{H} = \frac{1}{2} \int_{\mathbb{R}^N} \mathbf{u} \cdot \mathbf{m} d\mathbf{x}.$$

then  $\frac{\delta \mathcal{H}}{\delta \mathbf{m}} = \mathbf{u}$  and (EP) can be recast as

$$\partial_t \mathbf{m} = -\mathcal{A} \frac{\delta \mathcal{H}}{\delta \mathbf{m}}, \quad (6)$$

where  $\mathcal{A}$  is an anti-symmetric operator defined by

$$\mathcal{A} \mathbf{u} = \sum_{j=1}^N \partial_j (m_i u_j) + \sum_{j=1}^N \partial_i u_j m_j.$$

Consequently, from (2) and (6), there are two conservation laws

$$\frac{d}{dt} \int_{\mathbb{R}^N} \mathbf{m} d\mathbf{x} = 0, \quad \frac{d}{dt} \int_{\mathbb{R}^N} (|\mathbf{u}|^2 + \alpha |\nabla \mathbf{u}|^2) d\mathbf{x} = 0.$$

For the one-dimensional case, (EP) coincides with the dispersion-less case of Camassa-Holm (CH) equation and there is an additional Hamiltonian structure and a Lax-pair

which leads to a complete integrability of (CH) [4]. We refer to [13] for a general discussion on bi-Hamiltonian system and complete integrability.

When  $\alpha = 0$ , the above Hamiltonian structure shows that (EP) is a symmetric hyperbolic system of conservation laws

$$\begin{cases} \partial_t \mathbf{u} + \operatorname{div}(\mathbf{u} \otimes \mathbf{u}) + \frac{1}{2} \nabla |\mathbf{u}|^2 = 0 \\ \mathbf{u}(x, 0) = \mathbf{u}_0 \end{cases} \quad (7)$$

which possess a global convex entropy function

$$\frac{1}{2} \partial_t |\mathbf{u}|^2 + \operatorname{div}(|\mathbf{u}|^2 \mathbf{u}) = 0. \quad (8)$$

We refer (7) as the zero dispersion equation. Indeed, we can recast it in a usual form of a symmetric hyperbolic system (we state it in  $\mathbb{R}^3$ ):

$$\mathbf{u}_t + A \mathbf{u}_x + B \mathbf{u}_y + C \mathbf{u}_z = 0$$

with

$$\mathbf{u} = \begin{pmatrix} u \\ v \\ w \end{pmatrix}, A = \begin{pmatrix} 3u & v & w \\ v & u & 0 \\ w & 0 & u \end{pmatrix}, \dots$$

$A$  is a symmetric matrix and has three eigenvalues:  $u$ ,  $2u + |\mathbf{u}|$ ,  $2u - |\mathbf{u}|$ , corresponding to one linearly degenerate field, and two genuinely nonlinear fields, respectively, when  $\mathbf{u} \neq 0$ .

We shall remark that although the high dimensional Burgers equation has a similar structure as (7), it does not possess a global convex entropy. In section 5, we will prove a Liouville type theorem for (7). This theorem does not hold true for the high dimensional Burgers equation.

Now we introduce some notations and then we state a theorem on local existence of the weak solution and local existence and uniqueness of the classical solution.

For  $s \in \mathbb{R}$  and  $p \in [1, \infty]$  we define the Bessel potential space  $L^{s,p}(\mathbb{R}^N)$  as follows

$$L^{s,p}(\mathbb{R}^N) = \{f \in L^p(\mathbb{R}^N) \mid \|(1 - \Delta)^{\frac{s}{2}} f\|_{L^p} := \|f\|_{L^{s,p}} < \infty\}.$$

For  $s \in \mathbb{N} \cup \{0\}$  it is well-known that  $L^{s,p}(\mathbb{R}^N)$  is equivalent to the standard Sobolev space  $W^{s,p}(\mathbb{R}^N)$  (see e.g. [27]). This, in turn, implies immediately that there exist  $C_1, C_2$  such that

$$C_1 \|\mathbf{u}\|_{W^{k+2,p}} \leq \|\mathbf{m}\|_{L^{k,p}} \leq C_2 \|\mathbf{u}\|_{W^{k+2,p}} \quad (9)$$

for all  $k \in \mathbb{N} \cup \{0\}, p \in (1, \infty)$ . As usual we denote  $H^s(\mathbb{R}^N) = W^{s,2}(\mathbb{R}^N)$ .

**Theorem 1** *(i) Assume  $\alpha > 0$  and  $\mathbf{u}_0 \in W^{2,p}(\mathbb{R}^N)$  with  $p > N$ . Then, there exists  $T = T(\|\mathbf{u}_0\|_{W^{2,p}})$  such that a weak solution to (EP) exists, and belongs to  $\mathbf{u} \in L^\infty(0, T; W^{2,p}(\mathbb{R}^N)) \cap Lip(0, T; W^{1,p}(\mathbb{R}^N))$ .*

- (ii) Let  $\alpha > 0$  and  $\mathbf{u}_0 \in H^k(\mathbb{R}^N)$  with  $k > N/2 + 3$ . Then, there exists  $T = T(\|\mathbf{u}_0\|_{H^k})$  such that a classic solution to (EP) exists uniquely, and belongs to  $\mathbf{u} \in C([0, T]; H^k(\mathbb{R}^N))$ .
- (iii) For  $\alpha = 0$ , (EP) is a symmetric hyperbolic system of conservation laws with a convex entropy. Consequently, if  $\mathbf{u}_0 \in H^k(\mathbb{R}^N)$  with  $k > N/2 + 1$ . Then, there exists  $T = T(\|\mathbf{u}_0\|_{H^k})$  such that a classic solution to (EP) exists uniquely, and belongs to  $\mathbf{u} \in C([0, T]; H^k(\mathbb{R}^N))$ .

The proof of symmetric hyperbolicity and existence of convex entropy in (iii) are given in (7)-(8). The proof of existence of the unique classic solution for symmetric hyperbolic system is standard, see e.g [16]. The proof of (i) and (ii) is also rather standard and will be given in the Appendix for completeness.

### 3 Finite time blow up

In this section, we first present a theorem on a blow-up criterion and then we prove a theorem on finite time blow up for the zero dispersion equation.

We denote the deformation tensor for  $\mathbf{u}$  by  $S = (S_{ij})$ , where  $S_{ij} := \frac{1}{2}(\partial_i u_j + \partial_j u_i)$ . We recall the Besov space  $\dot{B}_{\infty, \infty}^0$ , which is defined as follows. Let  $\{\psi_m\}_{m \in \mathbb{Z}}$  be the Littlewood-Paley partition of unity, where the Fourier transform  $\hat{\psi}_m(\xi)$  is supported on the annulus  $\{\xi \in \mathbb{R}^N \mid 2^{m-1} \leq |\xi| < 2^m\}$  (see e.g. [28]). Then,

$$f \in \dot{B}_{\infty, \infty}^0 \quad \text{if and only if} \quad \sup_{m \in \mathbb{Z}} \|\psi_m * f\|_{L^\infty} := \|f\|_{\dot{B}_{\infty, \infty}^0} < \infty.$$

The following is a well-known embedding result,

$$L^\infty(\mathbb{R}^N) \hookrightarrow BMO(\mathbb{R}^N) \hookrightarrow \dot{B}_{\infty, \infty}^0(\mathbb{R}^N). \quad (10)$$

**Theorem 2** For  $\alpha \geq 0$ , we have the following finite time blow-up criterion of the local solution of (EP) in  $\mathbf{u} \in C([0, t_*]; H^k(\mathbb{R}^N))$ ,  $k > N/2 + 3$ .

$$\limsup_{t \rightarrow t_*} \|\mathbf{u}(t)\|_{H^k} = \infty \quad \text{if and only if} \quad \int_0^{t_*} \|S(t)\|_{\dot{B}_{\infty, \infty}^0} dt = \infty. \quad (11)$$

*Remark 1.1* Combining the embedding relation,  $W^{1, N}(\mathbb{R}^N) \hookrightarrow BMO(\mathbb{R}^N) \hookrightarrow \dot{B}_{\infty, \infty}^0(\mathbb{R}^N)$  with the inequality  $\|D^2 \mathbf{u}\|_{L^p} \leq C \|\mathbf{m}\|_{L^p}$  for  $p \in (1, \infty)$  (see (15) below), we have

$$\|S\|_{\dot{B}_{\infty, \infty}^0} \leq C \|S\|_{BMO} \leq C \|DS\|_{L^N} \leq C \|D^2 \mathbf{u}\|_{L^N} \leq C \|\mathbf{m}\|_{L^N}.$$

Therefore we obtain the following criterion as an immediate corollary of the above theorem: for all  $p > N$

$$\limsup_{t \rightarrow t_*} \|\mathbf{m}(t)\|_{L^p} = \infty \quad \text{if and only if} \quad \int_0^{t_*} \|\mathbf{m}(t)\|_{L^N} dt = \infty. \quad (12)$$

*Remark 1.2* In the one dimensional case of the Camassa-Holm equation (CH) the above criterion implies that finite time blow-up does not happen if  $\int_0^t \|\mathbf{u}_{xx}(\tau)\|_{L^1} d\tau < \infty$  for all  $t > 0$ . Thanks to the conservation law we have  $\sup_{0 < \tau < t} \|\mathbf{u}_x(\tau)\|_{L^2} \leq \|\mathbf{u}_0\|_{H^1} < \infty$  for all  $t > 0$ . Since we have embedding  $W^{2,1}(\mathbb{R}) \hookrightarrow H^1(\mathbb{R})$ , and we do have finite time blow-up for (CH) [24], our criterion is sharp in this one dimensional case.

*Proof of Theorem 2* We only give a proof for the case  $\alpha > 0$ . The proof for the case  $\alpha = 0$  is similar and simpler hence will be omitted.

Using estimates (33, 34, 35, 36) for  $I_1, I_2, I_3$  in the proof of Theorem 1 in the Appendix, one has

$$\begin{aligned} \frac{d}{dt} \|\mathbf{m}(t)\|_{H^k} &\leq C(\|\nabla \mathbf{u}\|_{L^\infty} + \|\mathbf{m}\|_{L^\infty} + \|\nabla \mathbf{m}\|_{L^\infty}) \|\mathbf{m}(t)\|_{H^k} \\ &\leq C(\|\mathbf{m}\|_{L^p} + \|D\mathbf{m}\|_{L^p} + \|D^2\mathbf{m}\|_{L^p}) \|\mathbf{m}(t)\|_{H^k}. \end{aligned}$$

Hence,

$$\|\mathbf{m}(t)\|_{H^k} \leq \|\mathbf{m}_0\|_{H^k} \exp \left[ C \int_0^t \{ \|\mathbf{m}(\tau)\|_{L^p} + \|D\mathbf{m}(\tau)\|_{L^p} + \|D^2\mathbf{m}(\tau)\|_{L^p} \} d\tau \right] \quad (13)$$

for  $k > N/2 + 1$  and  $p > N$ , where we used the Sobolev embedding. Consequently, blow up of  $\|\mathbf{m}(t)\|_{H^k}$  as  $t \rightarrow t^*$  implies that at least one of  $\|\mathbf{m}(t)\|_{L^p}$ ,  $\|D\mathbf{m}(t)\|_{L^p}$  and  $\|D^2\mathbf{m}(t)\|_{L^p}$  blow up as  $t \rightarrow t^*$ . In the following three steps, we show blow-up criterion for each of them are all given by (11).

*Step 1.* We first recall the following logarithmic Sobolev inequality (see e.g. [28]),

$$\|f\|_{L^\infty} \leq C(1 + \|f\|_{\dot{B}_{\infty,\infty}^0})(\log(1 + \|f\|_{W^{s,p}})), \quad (14)$$

where  $s > 0, 1 < p < \infty$  and  $sp > N$ . From the estimate in (31) in the Appendix we obtain

$$\begin{aligned} \frac{d}{dt} \|\mathbf{m}\|_{L^p} &\leq C(1 + \|S\|_{\dot{B}_{\infty,\infty}^0}) \log(1 + \|S\|_{W^{1,p}}) \|\mathbf{m}\|_{L^p} \quad (\text{for } p > N) \\ &\leq C(1 + \|S\|_{\dot{B}_{\infty,\infty}^0}) \log(1 + \|D^2\mathbf{u}\|_{L^p}) \|\mathbf{m}\|_{L^p} \\ &\leq C(1 + \|S\|_{\dot{B}_{\infty,\infty}^0}) \log(1 + \|\mathbf{m}\|_{L^p}) \|\mathbf{m}\|_{L^p} \end{aligned}$$

for  $p > N$ , where we used the boundedness on  $L^p(\mathbb{R}^N)$  of the pseudo-differential operator

$$\sigma_{ij}(D) := \partial_i \partial_j (1 - \alpha \Delta)^{-1} = -R_i R_j \Delta (1 - \alpha \Delta)^{-1}$$

with the Riesz transforms  $\{R_j\}_{j=1}^N$  on  $\mathbb{R}^N$  (see Lemma 2.1, pp. 133[27]), which provides us with

$$\|D^2\mathbf{u}\|_{L^p} = \sum_{i,j=1}^N \|\sigma_{ij}(D)\mathbf{m}\|_{L^p} \leq C \|\mathbf{m}\|_{L^p} \quad (15)$$

for all  $p \in (1, \infty)$ . By Gronwall's lemma we obtain

$$\log(1 + \|\mathbf{m}(t)\|_{L^p}) \leq \log(1 + \|\mathbf{m}_0\|_{L^p}) \exp\left(C \int_0^t (1 + \|S(\tau)\|_{\dot{B}_{\infty,\infty}^0}) d\tau\right) \quad (16)$$

for  $p > N$ . This implies that

$$\limsup_{t \rightarrow t_*} \|\mathbf{m}(t)\|_{L^p} = \infty \quad \text{if and only if} \quad \int_0^{t_*} \|S(t)\|_{\dot{B}_{\infty,\infty}^0} dt = \infty. \quad (17)$$

*Step 2.* Taking derivative of (EP) and  $L^2(\mathbb{R}^N)$  inner product it with  $D\mathbf{m}|D\mathbf{m}|^{p-2}$ , we find that

$$\begin{aligned} \frac{1}{p} \frac{d}{dt} \|D\mathbf{m}(t)\|_{L^p}^p &= \frac{1}{p} \int_{\mathbb{R}^N} (\operatorname{div} \mathbf{u}) |D\mathbf{m}|^p dx - \int_{\mathbb{R}^N} (D\mathbf{u} \cdot \nabla) \mathbf{m} \cdot D\mathbf{m} |D\mathbf{m}|^{p-2} dx \\ &\quad - \int_{\mathbb{R}^N} D(\nabla \mathbf{u})^\top \mathbf{m} \cdot D\mathbf{m} |D\mathbf{m}|^{p-2} dx - \int_{\mathbb{R}^N} (\nabla \mathbf{u})^\top D\mathbf{m} \cdot D\mathbf{m} |D\mathbf{m}|^{p-2} dx \\ &\quad - \int_{\mathbb{R}^N} D(\operatorname{div} \mathbf{u}) \mathbf{m} \cdot D\mathbf{m} |D\mathbf{m}|^{p-2} dx - \int_{\mathbb{R}^N} (\operatorname{div} \mathbf{u}) D\mathbf{m} \cdot D\mathbf{m} |D\mathbf{m}|^{p-2} dx \\ &\leq \left(3 + \frac{1}{p}\right) \int_{\mathbb{R}^N} |D\mathbf{u}| |D\mathbf{m}|^p dx + 2 \int_{\mathbb{R}^N} |D^2 \mathbf{u}| |\mathbf{m}| |D\mathbf{m}|^{p-1} dx \\ &\leq \left(3 + \frac{1}{p}\right) \|D\mathbf{u}\|_{L^\infty} \|D\mathbf{m}\|_{L^p}^p + 2 \|D^2 \mathbf{u}\|_{L^{2p}} \|\mathbf{m}\|_{L^{2p}} \|D\mathbf{m}\|_{L^p}^{p-1} \\ &\leq C \|\mathbf{m}\|_{L^p} \|D\mathbf{m}\|_{L^p}^p + C \|\mathbf{m}\|_{L^{2p}}^2 \|D\mathbf{m}\|_{L^p}^{p-1} \end{aligned}$$

for  $p > N$ , where we used the Sobolev embedding and (15) to estimate

$$\|D\mathbf{u}\|_{L^\infty} \leq C \|D^2 \mathbf{u}\|_{L^p} \leq C \|\mathbf{m}\|_{L^p}$$

for  $p > N$ . Hence, for  $p > N$  we have

$$\frac{d}{dt} \|D\mathbf{m}(t)\|_{L^p} \leq C \|\mathbf{m}\|_{L^p} \|D\mathbf{m}\|_{L^p} + C \|\mathbf{m}\|_{L^{2p}}^2.$$

By Gronwall's lemma, we have

$$\|D\mathbf{m}(t)\|_{L^p} \leq \exp\left(C \int_0^t \|\mathbf{m}(\tau)\|_{L^p} d\tau\right) \left(\|D\mathbf{m}_0\|_{L^p} + C \int_0^t \|\mathbf{m}(\tau)\|_{L^{2p}}^2 d\tau\right) \quad (18)$$

for  $p > N$ . From estimate (16), one has

$$\begin{aligned} \int_0^t \|\mathbf{m}(s)\|_{L^p} ds &\leq t \max_{0 \leq s \leq t} \|\mathbf{m}(s)\|_{L^p} \\ &\leq t \max_{0 \leq s \leq t} \exp(\log(1 + \|\mathbf{m}(s)\|_{L^p})) \\ &\leq t \exp\left(\log(1 + \|\mathbf{m}_0\|_{L^p}) \exp\left(C \int_0^t (1 + \|S(\tau)\|_{\dot{B}_{\infty,\infty}^0}) d\tau\right)\right). \end{aligned} \quad (19)$$



Similarly,

$$\int_0^t \|\mathbf{m}(s)\|_{L^{2p}} ds \leq t \exp \left( \log(1 + \|\mathbf{m}_0\|_{L^{2p}}) \exp \left( C \int_0^t (1 + \|S(\tau)\|_{\dot{B}_{\infty,\infty}^0}) d\tau \right) \right). \quad (20)$$

Combining (18, 19) and (20), one obtains

$$\limsup_{t \rightarrow t_*} \|D\mathbf{m}(t)\|_{L^p} = \infty \quad \text{if and only if} \quad \int_0^{t_*} \|S(t)\|_{\dot{B}_{\infty,\infty}^0} dt = \infty. \quad (21)$$

*Step 3.* Similarly, taking  $D^2$  of (EP) and  $L^2(\mathbb{R}^N)$  inner product it with  $D^2\mathbf{m}|D^2\mathbf{m}|^{p-2}$ , we find that

$$\begin{aligned} \frac{1}{p} \frac{d}{dt} \|D^2\mathbf{m}(t)\|_{L^p}^p &\leq 4 \int_{\mathbb{R}^N} |D\mathbf{u}| |D^2\mathbf{m}|^p dx + 3 \int_{\mathbb{R}^N} |D^2\mathbf{u}| |D\mathbf{m}| |D^2\mathbf{m}|^{p-1} dx \\ &\quad + 2 \int_{\mathbb{R}^N} |D^3\mathbf{u}| |\mathbf{m}| |D^2\mathbf{m}|^{p-1} dx \\ &\leq 4 \|D\mathbf{u}\|_{L^\infty} \|D^2\mathbf{m}\|_{L^p}^p + 3 \|D^2\mathbf{u}\|_{L^{2p}} \|D\mathbf{m}\|_{L^{2p}} \|D^2\mathbf{m}\|_{L^p}^{p-1} \\ &\quad + 2 \|D^3\mathbf{u}\|_{L^{2p}} \|\mathbf{m}\|_{L^{2p}} \|D^2\mathbf{m}\|_{L^p}^{p-1} \\ &\leq C \|\mathbf{m}\|_{L^p} \|D^2\mathbf{m}\|_{L^p}^p + C \|\mathbf{m}\|_{L^{2p}} \|D\mathbf{m}\|_{L^{2p}} \|D^2\mathbf{m}\|_{L^p}^{p-1} \end{aligned}$$

for  $p > N$ , where we used the estimate (15) as follows

$$\begin{aligned} \|D^3\mathbf{u}\|_{L^q} &= \|\{D^2(1 - \alpha\Delta)^{-1}\} D(1 - \alpha\Delta)\mathbf{u}\|_{L^q} \\ &\leq \sum_{i,j=1}^N \|\sigma_{ij}(D)D\mathbf{m}\|_{L^q} \leq C \|D\mathbf{m}\|_{L^q}, \end{aligned}$$

which holds for all  $q \in (1, \infty)$ . Hence,

$$\frac{d}{dt} \|D^2\mathbf{m}(t)\|_{L^p} \leq C \|\mathbf{m}\|_{L^p} \|D^2\mathbf{m}\|_{L^p} + C \|\mathbf{m}\|_{L^{2p}} \|D\mathbf{m}\|_{L^{2p}}.$$

By Gronwall's lemma we have

$$\|D^2\mathbf{m}(t)\|_{L^p} \leq \exp \left( C \int_0^t \|\mathbf{m}(\tau)\|_{L^p} d\tau \right) \left( \|D^2\mathbf{m}_0\|_{L^p} + C \int_0^t \|\mathbf{m}(\tau)\|_{L^{2p}} \|D\mathbf{m}(\tau)\|_{L^{2p}} d\tau \right)$$

for  $p > N$ . Similarly to the estimates in (19) and (20), the right hand side terms in the above inequality can all be controlled

$$\int_0^t (1 + \|S(\tau)\|_{\dot{B}_{\infty,\infty}^0}) d\tau.$$

Therefore, we have

$$\limsup_{t \rightarrow t_*} \|D^2\mathbf{m}(t)\|_{L^p} = \infty \quad \text{if and only if} \quad \int_0^{t_*} \|S(t)\|_{\dot{B}_{\infty,\infty}^0} dt = \infty. \quad (22)$$

Combination of (13, 17, 21, 22) gives the proof of the theorem.  $\square$

We now present a finite time blow-up result for  $\alpha = 0$ .

**Theorem 3** *Let the initial data of the system (7),  $\mathbf{u}_0 \in H^k(\mathbb{R}^N)$ ,  $k > N/2 + 2$ , has the reflection symmetry with respect to the origin, and satisfies  $\operatorname{div} \mathbf{u}_0(0) < 0$ . Then, there exists a finite time blow-up of the classical solution.*

*Proof* Taking divergence of (7), we find

$$\partial_t(\operatorname{div} \mathbf{u}) + \mathbf{u} \cdot \nabla(\operatorname{div} \mathbf{u}) + 2 \sum_{i,j=1}^N S_{ij}^2 + \sum_{j=1}^N (\Delta u_j) u_j + (\operatorname{div} \mathbf{u})^2 + \sum_{i,j=1}^N (\partial_i \partial_j u_i) u_j = 0, \quad (23)$$

where we used  $S_{ij} = \frac{1}{2}(\partial_i u_j + \partial_j u_i)$ , and the fact

$$\sum_{i,j=1}^N \partial_i u_j \partial_j u_i + \sum_{i,j=1}^N \partial_i u_j \partial_i u_j = 2 \sum_{i,j=1}^N \partial_i u_j S_{ij} = \sum_{i,j=1}^N (\partial_i u_j + \partial_j u_i) S_{ij} = 2 \sum_{i,j=1}^N S_{ij}^2.$$

Now we consider the reflection transform:

$$R : x \rightarrow \bar{x} = -x, \quad \mathbf{u}(x, t) \rightarrow \bar{\mathbf{u}}(x, t) = -\mathbf{u}(-x, t).$$

Obviously the system (7) is invariant under this transform. The origin of the coordinate is the invariant point under the reflection transform. We consider the smooth initial data  $\mathbf{u}_0 \in H^k(\mathbb{R}^N)$ ,  $k > N/2 + 2$ , which has the reflection symmetry. Then, by the uniqueness of the local classical solution in  $H^k(\mathbb{R}^N)$ , and hence in  $C^2(\mathbb{R}^N)$ , the reflection symmetry is preserved as long as the classical solution persists. We consider the evolution of the solution at the origin of the coordinates. Then,  $\mathbf{u}(0, t) = 0$  and  $D^2 \mathbf{u}(0, t) = 0$  for all  $t \in [0, T_*)$ , where  $T_*$  is the maximal time of existence of the classical solution in  $H^k(\mathbb{R}^N)$ . If  $T_* = \infty$ , we will show that this leads to a contradiction. The system (23) at the origin is reduced to

$$\partial_t(\operatorname{div} \mathbf{u}) + 2 \sum_{i,j=1}^N S_{ij}^2 + (\operatorname{div} \mathbf{u})^2 = 0,$$

which implies

$$\partial_t(\operatorname{div} \mathbf{u}) = -2 \sum_{i,j=1}^N S_{ij}^2 - (\operatorname{div} \mathbf{u})^2 \leq -(\operatorname{div} \mathbf{u})^2. \quad (24)$$

Thus,

$$\operatorname{div} \mathbf{u}(0, t) \leq \frac{\operatorname{div} \mathbf{u}_0(0)}{1 + \operatorname{div} \mathbf{u}_0(0)t},$$

which shows  $T_* \leq \frac{1}{|\operatorname{div} \mathbf{u}_0(0)|}$  for  $\operatorname{div} \mathbf{u}_0(0) < 0$ .  $\square$

## 4 Zero $\alpha$ limit for weak solutions

In this section, we show the following theorem on the zero dispersion limit  $\alpha \rightarrow 0$  for the weak solutions.

**Theorem 4** Let  $\mathbf{u}^\alpha \in L^\infty((0, T); H^1(\mathbb{R}^N))$  be a weak solution with initial data  $\mathbf{u}_0^\alpha$  to (EP) with  $\alpha > 0$ , and  $\mathbf{u} \in L^\infty((0, T); H^k(\mathbb{R}^N)) \cap Lip((0, T); H^2(\mathbb{R}^N))$ ,  $k > N/2 + 3$ , be the classic solution with initial data  $\mathbf{u}_0$  to (EP) with  $\alpha = 0$ , i.e., (7). Then, we have

$$\begin{aligned} & \sup_{0 \leq t \leq T} \{ \|\mathbf{u}^\alpha(t) - \mathbf{u}(t)\|_{L^2} + \sqrt{\alpha} \|\nabla(\mathbf{u}^\alpha(t) - \mathbf{u}(t))\|_{L^2} \} \\ & \leq C (\alpha + \|\mathbf{u}_0^\alpha - \mathbf{u}_0\|_{L^2} + \sqrt{\alpha} \|\nabla(\mathbf{u}_0^\alpha - \mathbf{u}_0)\|_{L^2}), \end{aligned} \quad (25)$$

where  $C = C(\|\mathbf{u}\|_{L^\infty(0, T; H^k(\mathbb{R}^N))}, \|\mathbf{u}\|_{Lip(0, T; H^2(\mathbb{R}^N))})$  is a constant.

*Proof* We denote  $\mathbf{m} := \mathbf{u} - \alpha \Delta \mathbf{u}$ . Then  $(\mathbf{u}, \mathbf{m})$  satisfy (EP) with a truncation term as below

$$\partial_t \mathbf{m} + \operatorname{div}(\mathbf{u} \otimes \mathbf{m}) + (\nabla \mathbf{u})^\top \mathbf{m} = -\alpha \{ \Delta \mathbf{u}_t + \operatorname{div}(\mathbf{u} \otimes \Delta \mathbf{u}) + (\nabla \mathbf{u})^\top \Delta \mathbf{u} \}. \quad (26)$$

Subtracting (26) from the first equation of (EP), and setting  $\bar{\mathbf{m}} := \mathbf{m}^\alpha - \mathbf{m}$  and  $\bar{\mathbf{u}} := \mathbf{u}^\alpha - \mathbf{u}$ , we find

$$\begin{aligned} & \partial_t \bar{\mathbf{m}} + \operatorname{div}(\bar{\mathbf{u}} \otimes \bar{\mathbf{m}}) + \operatorname{div}(\bar{\mathbf{u}} \otimes \mathbf{m}) + \operatorname{div}(\mathbf{u} \otimes \bar{\mathbf{m}}) + (\nabla \bar{\mathbf{u}})^\top \bar{\mathbf{m}} + (\nabla \bar{\mathbf{u}})^\top \mathbf{m} + (\nabla \mathbf{u})^\top \bar{\mathbf{m}} \\ & = \alpha \{ \Delta \mathbf{u}_t + \operatorname{div}(\mathbf{u} \otimes \Delta \mathbf{u}) + (\nabla \mathbf{u})^\top \Delta \mathbf{u} \} \end{aligned} \quad (27)$$

Taking  $L^2(\mathbb{R}^N)$  inner product (27) with  $\bar{\mathbf{u}}$ , and integrating by part, and observing

$$\begin{aligned} & \int_{\mathbb{R}^N} \operatorname{div}(\bar{\mathbf{u}} \otimes \bar{\mathbf{m}}) \cdot \bar{\mathbf{u}} \, dx = - \int_{\mathbb{R}^N} \bar{\mathbf{u}} \cdot (\nabla \bar{\mathbf{u}})^\top \bar{\mathbf{m}} \, dx \\ & \int_{\mathbb{R}^N} \operatorname{div}(\bar{\mathbf{u}} \otimes \mathbf{m}) \cdot \bar{\mathbf{u}} \, dx = - \int_{\mathbb{R}^N} \bar{\mathbf{u}} \cdot (\nabla \bar{\mathbf{u}})^\top \mathbf{m} \, dx, \end{aligned}$$

we obtain that

$$\begin{aligned} & \frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^N} (|\bar{\mathbf{u}}|^2 + \alpha |\nabla \bar{\mathbf{u}}|^2) \, dx = - \int_{\mathbb{R}^N} \operatorname{div}(\mathbf{u} \otimes \bar{\mathbf{m}}) \cdot \bar{\mathbf{u}} \, dx - \int_{\mathbb{R}^N} \bar{\mathbf{u}} \cdot (\nabla \mathbf{u})^\top \bar{\mathbf{m}} \, dx \\ & \quad + \alpha \int_{\mathbb{R}^N} [\bar{\mathbf{u}} \cdot \{ \Delta \mathbf{u}_t + \operatorname{div}(\mathbf{u} \otimes \Delta \mathbf{u}) + (\nabla \mathbf{u})^\top \Delta \mathbf{u} \}] \, dx \\ & := I_1 + I_2 + I_3. \end{aligned}$$

We estimate

$$\begin{aligned} I_1 &= - \sum_{i,j=1}^N \int_{\mathbb{R}^N} \partial_i u_i (\bar{u}_j - \alpha \Delta \bar{u}_j) \bar{u}_j \, dx - \sum_{i,j=1}^N \int_{\mathbb{R}^N} u_i \partial_i (\bar{u}_j - \alpha \Delta \bar{u}_j) \bar{u}_j \, dx \\ &= J_1 + J_2, \end{aligned}$$

where

$$\begin{aligned} J_1 &= - \sum_{i,j=1}^N \int_{\mathbb{R}^N} \partial_i u_i |\bar{u}_j|^2 \, dx + \alpha \sum_{i,j,k=1}^N \int_{\mathbb{R}^N} \partial_i \partial_k u_i (\partial_k \bar{u}_j) \bar{u}_j \, dx \\ & \quad + \alpha \sum_{i,j,k=1}^N \int_{\mathbb{R}^N} \partial_i u_i (\partial_k \bar{u}_j) \partial_k \bar{u}_j \, dx \\ & \leq C \|\mathbf{u}(t)\|_{C^2} (\|\bar{\mathbf{u}}\|_{L^2}^2 + \alpha \|\nabla \bar{\mathbf{u}}\|_{L^2}^2), \end{aligned}$$

and

$$\begin{aligned}
J_2 &= - \sum_{i,j=1}^N \int_{\mathbb{R}^N} u_i (\partial_i \bar{u}_j) \bar{u}_j \, dx + \alpha \sum_{i,j=1}^N \int_{\mathbb{R}^N} u_i \partial_i (\Delta \bar{u}_j) \bar{u}_j \, dx \\
&= - \frac{1}{2} \sum_{i,j=1}^N \int_{\mathbb{R}^N} u_i \partial_i |\bar{u}_j|^2 \, dx - \alpha \sum_{i,j,k=1}^N \int_{\mathbb{R}^N} \partial_k u_i \partial_i (\partial_k \bar{u}_j) \bar{u}_j \, dx \\
&\quad - \frac{\alpha}{2} \sum_{i,j,k=1}^N \int_{\mathbb{R}^N} u_i \partial_i |\partial_k \bar{u}_j|^2 \, dx \\
&= \frac{1}{2} \sum_{i,j=1}^N \int_{\mathbb{R}^N} \partial_i u_i |\bar{u}_j|^2 \, dx + \alpha \sum_{i,j,k=1}^N \int_{\mathbb{R}^N} \partial_i \partial_k u_i (\partial_k \bar{u}_j) \bar{u}_j \, dx \\
&\quad + \alpha \sum_{i,j,k=1}^N \int_{\mathbb{R}^N} \partial_k u_i (\partial_k \bar{u}_j) \partial_i \bar{u}_j \, dx + \frac{\alpha}{2} \sum_{i,j,k=1}^N \int_{\mathbb{R}^N} \partial_i u_i |\partial_k \bar{u}_j|^2 \, dx \\
&\leq C \|u(t)\|_{C^2} (\|\bar{\mathbf{u}}\|_{L^2}^2 + \alpha \|\nabla \bar{\mathbf{u}}\|_{L^2}^2).
\end{aligned}$$

$$\begin{aligned}
I_2 &= \sum_{i,j=1}^N \int_{\mathbb{R}^N} \bar{u}_i \partial_i u_j (\bar{u}_j - \alpha \Delta \bar{u}_j) \, dx \\
&= \sum_{i,j=1}^N \int_{\mathbb{R}^N} \bar{u}_i \partial_i u_j \bar{u}_j \, dx + \alpha \sum_{i,j,k=1}^N \int_{\mathbb{R}^N} \partial_k \bar{u}_i \partial_i u_j \partial_k \bar{u}_j \, dx \\
&\quad + \alpha \sum_{i,j,k=1}^N \int_{\mathbb{R}^N} \bar{u}_i \partial_i \partial_k u_j \partial_k \bar{u}_j \, dx \\
&\leq C \|u(t)\|_{C^2} (\|\bar{\mathbf{u}}\|_{L^2}^2 + \alpha \|\nabla \bar{\mathbf{u}}\|_{L^2}^2).
\end{aligned}$$

One can estimate  $I_3$  immediately as

$$I_3 \leq \|\bar{\mathbf{u}}\|_{L^2}^2 + \alpha^2 C (\|\mathbf{u}\|_{Lip(0,T;H^2(\mathbb{R}^N))}^2 + \|\mathbf{u}\|_{L^\infty(0,T;H^3(\mathbb{R}^N))}^4).$$

Summarizing the above estimates, we obtain

$$\frac{d}{dt} (\|\bar{\mathbf{u}}\|_{L^2}^2 + \alpha \|\nabla \bar{\mathbf{u}}\|_{L^2}^2) \leq C \|u(t)\|_{C^2} (\|\bar{\mathbf{u}}\|_{L^2}^2 + \alpha \|\nabla \bar{\mathbf{u}}\|_{L^2}^2) + \alpha^2 C (\|\mathbf{u}\|_{Lip(0,T;H^2(\mathbb{R}^N))}^2 + \|\mathbf{u}\|_{L^\infty(0,T;H^3(\mathbb{R}^N))}^4),$$

which implies by Gronwall's lemma that

$$\|\bar{\mathbf{u}}\|_{L^2}^2 + \alpha \|\nabla \bar{\mathbf{u}}\|_{L^2}^2 \leq C_1 (\alpha^2 + \|\bar{\mathbf{u}}(0)\|_{L^2}^2 + \alpha \|\nabla \bar{\mathbf{u}}(0)\|_{L^2}^2)$$

where constant  $C_1$  depended only on  $\|\mathbf{u}\|_{Lip(0,T;H^2(\mathbb{R}^N))}$  and  $\|\mathbf{u}\|_{L^\infty(0,T;H^3(\mathbb{R}^N))}$ . This completes the proof of theorem.  $\square$

## 5 Liouville type theorem for stationary solutions

In this section, we prove a Liouville type theorem for stationary solutions. Recall that the stationary weak solution defined in Definition 1 reduces to

**Definition 2**  $\mathbf{u} \in H^1(\mathbb{R}^N)$  is a stationary weak solution to (EP) on  $\mathbb{R}^N$ , if the following holds

$$\begin{aligned} & \sum_{j=1}^N \int_{\mathbb{R}^N} \{u_i u_j + \alpha \nabla u_i \cdot \nabla u_j\} \partial_j \varphi_i dx + \alpha \sum_{j=1}^N \int_{\mathbb{R}^N} u_j \nabla u_i \cdot \nabla \partial_j \varphi_i dx \\ & + \sum_{j=1}^N \int_{\mathbb{R}^N} \left\{ \frac{\delta_{ij}}{2} |\mathbf{u}|^2 - \alpha \partial_i \mathbf{u} \cdot \partial_j \mathbf{u} + \frac{\alpha \delta_{ij}}{2} |\nabla \mathbf{u}|^2 \right\} \partial_j \varphi_i dx = 0 \end{aligned} \quad (28)$$

for  $i = 1, \dots, N$  and for all  $\phi \in C_0^\infty(\mathbb{R}^N)$ .

**Theorem 5** (i) Let  $\mathbf{u} \in H^1(\mathbb{R}^N)$  be a stationary weak solution to (EP) with  $\alpha > 0$ . Then,  $\mathbf{u} = 0$ .

(ii) Let  $\mathbf{u} \in L^2(\mathbb{R}^N)$  be a stationary weak solution to (EP) with  $\alpha = 0$ . Then,  $\mathbf{u} = 0$ .

*Proof* For  $\alpha > 0$ , one can write (28) in the following form,

$$\sum_{j=1}^N \int_{\mathbb{R}^N} T_{ij}^a \partial_j \varphi_i dx + \sum_{j,k=1}^N \int_{\mathbb{R}^N} \tilde{T}_{ijk}^b \partial_j \partial_k \varphi_i dx = 0, \quad (29)$$

where  $T_{ij}^a$  is defined in (3) and we recall here

$$T_{ij}^a = u_i u_j + \alpha \nabla u_i \cdot \nabla u_j + \frac{\delta_{ij}}{2} |\mathbf{u}|^2 - \alpha \partial_i \mathbf{u} \cdot \partial_j \mathbf{u} + \frac{\alpha \delta_{ij}}{2} |\nabla \mathbf{u}|^2,$$

and

$$\tilde{T}_{ijk}^b = \alpha u_j \partial_k u_i.$$

corresponding to  $T_{ij}^b$  in (4).

Let us consider the radial cut-off function  $\sigma \in C_0^\infty(\mathbb{R}^N)$  such that

$$\sigma(|x|) = \begin{cases} 1 & \text{if } |x| < 1, \\ 0 & \text{if } |x| > 2, \end{cases}$$

and  $0 \leq \sigma(x) \leq 1$  for  $1 < |x| < 2$ . Then, for each  $R > 0$ , we define

$$\sigma\left(\frac{|x|}{R}\right) := \sigma_R(|x|) \in C_0^\infty(\mathbb{R}^N).$$

Choosing  $\varphi_i(x) = x_i \sigma_R(x)$  in (29), we obtain

$$\begin{aligned}
0 &= \sum_{i=1}^N \int_{\mathbb{R}^N} T_{ii}^a \sigma_R(x) dx + \sum_{i,j=1}^N \int_{\mathbb{R}^N} T_{ij}^a x_j \partial_i \sigma_R(x) dx + \sum_{i,k=1}^N \int_{\mathbb{R}^N} \tilde{T}_{ik}^b \partial_k \sigma_R(x) dx \\
&\quad + \sum_{i,j=1}^N \int_{\mathbb{R}^N} \tilde{T}_{ij}^b \partial_j \sigma_R(x) dx + \sum_{i,j,k=1}^N \int_{\mathbb{R}^N} \tilde{T}_{ijk}^b x_i \partial_j \partial_k \sigma_R(x) dx \\
&= I_1 + I_2 + I_3 + I_4 + I_5.
\end{aligned} \tag{30}$$

The hypothesis  $u \in H^1(\mathbb{R}^N)$  implies that  $T \in L^1(\mathbb{R}^N)$ . Thus, we obtain

$$|I_2| \leq \frac{1}{R} \int_{\{R \leq |x| \leq 2R\}} |T^a| |x| |\nabla \sigma| d\mathbf{x} \leq 2 \|\nabla \sigma\|_{L^\infty} \int_{\{R \leq |x| \leq 2R\}} |T^a| d\mathbf{x} \rightarrow 0$$

as  $R \rightarrow \infty$  by the dominated convergence theorem. Similarly,  $I_3, I_4, I_5 \rightarrow 0$  as  $R \rightarrow \infty$ .

Thus, passing  $R \rightarrow \infty$  in (30), we have

$$\begin{aligned}
0 &= \lim_{R \rightarrow \infty} \sum_{i=1}^N \int_{\mathbb{R}^N} T_{ii}^a \sigma_R(x) d\mathbf{x} \\
&= \int_{\mathbb{R}^N} \left\{ \frac{(N+2)}{2} |\mathbf{u}|^2 + \frac{\alpha N}{2} |\nabla \mathbf{u}|^2 \right\} d\mathbf{x},
\end{aligned}$$

which implies  $\mathbf{u} = 0$ . This gives (i).

For the case  $\alpha = 0$ . All the terms involving  $\alpha$  drop and (ii) holds true. This completes the proof the theorem  $\square$

We remark that the Liouville type results in Theorem 5 is rather surprising, as all the previous Liouville type results are for dissipative systems. For the Liouville type results for the dissipative systems, see, e.g. [5]. Theorem 5 is the first Liouville type theorem for non-dissipative systems.

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## Appendix: proof of Theorem 1

*Proof of Theorem 1* The proof of local existence part is standard, and below we derive the key local in time estimate of  $\mathbf{u}(t) \in L^\infty([0, T]; W^{2,p}(\mathbb{R}^N)) \cap Lip(0, T; W^{1,p}(\mathbb{R}^N))$ .

$$\begin{aligned}
\frac{1}{p} \frac{d}{dt} \|\mathbf{m}\|_{L^p}^p &= \frac{1}{p} \int_{\mathbb{R}^N} (\mathbf{u} \cdot \nabla) |\mathbf{m}|^p dx + \sum_{i,j=1}^N \int_{\mathbb{R}^N} \partial_j u_i m_i m_j |\mathbf{m}|^{p-2} dx \\
&\quad + \int_{\mathbb{R}^N} (\operatorname{div} \mathbf{u}) |\mathbf{m}|^p dx \\
&= \left(1 - \frac{1}{p}\right) \int_{\mathbb{R}^N} \operatorname{Tr}(S) |\mathbf{m}|^p dx + \sum_{i,j=1}^N \int_{\mathbb{R}^N} S_{ij} m_i m_j |\mathbf{m}|^{p-2} dx \\
&\leq C \|S\|_{L^\infty} \|\mathbf{m}\|_{L^p}^p \leq C \|\nabla \mathbf{u}\|_{L^\infty} \|\mathbf{m}\|_{L^p}^p \leq C \|\mathbf{m}\|_{L^p}^{p+1}, \tag{31}
\end{aligned}$$

and therefore

$$\frac{d}{dt} \|\mathbf{m}\|_{L^p} \leq C \|\mathbf{m}\|_{L^p}^2.$$

We thus have the following estimate on  $L^\infty(0, T; W^{2,p}(\mathbb{R}^N))$ ,

$$\|\mathbf{u}(t)\|_{W^{2,p}} \leq \frac{C \|\mathbf{u}_0\|_{W^{2,p}}}{1 - Ct \|\mathbf{u}_0\|_{W^{2,p}}} \quad \forall t \in [0, T], \tag{32}$$

where  $T = \frac{1}{\|\mathbf{u}_0\|_{W^{2,p}}}$ . In order to have estimate of  $\mathbf{u}$  in  $Lip(0, T; W^{1,p}(\mathbb{R}^N))$ , we take  $L^2(\mathbb{R}^N)$  inner product (EP) with the test function  $\psi \in W^{1, \frac{p}{p-1}}(\mathbb{R}^N)$  for  $p > N$ . Then,

$$\begin{aligned}
\int_{\mathbb{R}^N} \partial_t \mathbf{m} \cdot \psi dx &= \int_{\mathbb{R}^N} \mathbf{m} (\mathbf{u} \cdot \nabla) \psi dx - \int_{\mathbb{R}^N} \mathbf{m} \cdot \nabla \mathbf{u} \cdot \psi dx \\
&\leq C \|\mathbf{m}\|_{L^p} \|\mathbf{u}\|_{L^\infty} \|\nabla \psi\|_{L^{\frac{p}{p-1}}} + C \|\mathbf{m}\|_{L^p} \|\nabla \mathbf{u}\|_{L^\infty} \|\nabla \psi\|_{L^{\frac{p}{p-1}}} \\
&\leq C \|\mathbf{m}\|_{L^p}^2 \|\psi\|_{W^{1, \frac{p}{p-1}}},
\end{aligned}$$

which provides us with the estimate,

$$\|\partial_t \mathbf{u}\|_{L^\infty(0, T; W^{1,p}(\mathbb{R}^N))} \leq C \|\partial_t \mathbf{m}\|_{L^\infty(0, T; W^{-1,p}(\mathbb{R}^N))} \leq C \|\mathbf{m}\|_{L^\infty(0, T; L^p(\mathbb{R}^N))}^2.$$

Hence, for all  $0 < t_1 < t_2 < T$  we have

$$\|\mathbf{u}(t_2) - \mathbf{u}(t_1)\|_{W^{1,p}} \leq \int_{t_1}^{t_2} \|\partial_t \mathbf{u}(t)\|_{W^{1,p}} dt \leq C(t_2 - t_1) \|\mathbf{m}\|_{L^\infty(0, T; L^p(\mathbb{R}^N))}^2.$$

Namely,

$$\|\mathbf{u}\|_{Lip(0, T; W^{1,p}(\mathbb{R}^N))} \leq C \|\mathbf{m}\|_{L^\infty(0, T; L^p(\mathbb{R}^N))}^2.$$

This gives (i). Next we prove local in time persistency of regularity for  $\mathbf{u}(t)$  in  $H^k(\mathbb{R}^N)$  with  $k > N/2 + 3$ . Let  $\beta = (\beta_1, \dots, \beta_N)$  be the standard multi-index

notation with  $|\beta| = \beta_1 + \dots + \beta_N$ . Taking  $H^k(\mathbb{R}^N)$  inner product (EP) with  $\mathbf{m}$ , we find

$$\begin{aligned}
\frac{1}{2} \frac{d}{dt} \sum_{|\beta| \leq k} \|D^\beta \mathbf{m}\|_{L^2}^2 &= - \sum_{|\beta| \leq k} \int_{\mathbb{R}^N} D^\beta \{(\mathbf{u} \cdot \nabla) \mathbf{m}\} \cdot D^\beta \mathbf{m} \, dx \\
&\quad - \sum_{|\beta| \leq k} \int_{\mathbb{R}^N} D^\beta \{(\nabla \mathbf{u})^\top \mathbf{m}\} \cdot D^\beta \mathbf{m} \, dx \\
&\quad - \sum_{|\beta| \leq k} \int_{\mathbb{R}^N} D^\beta \{(\operatorname{div} \mathbf{u}) \mathbf{m}\} \cdot D^\beta \mathbf{m} \, dx \\
&:= I_1 + I_2 + I_3.
\end{aligned} \tag{33}$$

We write

$$\begin{aligned}
I_1 &= - \sum_{|\beta| \leq k} \int_{\mathbb{R}^N} \{D^\beta (\mathbf{u} \cdot \nabla) \mathbf{m} - (\mathbf{u} \cdot \nabla) D^\beta \mathbf{m}\} \cdot D^\beta \mathbf{m} \, dx \\
&\quad + \sum_{|\beta| \leq k} \int_{\mathbb{R}^N} (\mathbf{u} \cdot \nabla) D^\beta \mathbf{m} \cdot D^\beta \mathbf{m} \, dx \\
&:= J_1 + J_2,
\end{aligned}$$

and using the standard commutator estimate, we deduce

$$\begin{aligned}
J_1 &\leq \sum_{|\beta| \leq k} \|D^\beta (\mathbf{u} \cdot \nabla) \mathbf{m} - (\mathbf{u} \cdot \nabla) D^\beta \mathbf{m}\|_{L^2} \|D^\beta \mathbf{m}\|_{L^2} \\
&\leq C(\|\nabla \mathbf{u}\|_{L^\infty} \|\mathbf{m}\|_{H^k} + \|\mathbf{u}\|_{H^k} \|\nabla \mathbf{m}\|_{L^\infty}) \|\mathbf{m}\|_{H^k} \\
&\leq C(\|\mathbf{u}\|_{H^{N/2+1+\varepsilon}} \|\mathbf{m}\|_{H^k} + \|\mathbf{m}\|_{H^{k-2}} \|\mathbf{m}\|_{H^{N/2+1+\varepsilon}}) \|\mathbf{m}\|_{H^k} \quad (\forall \varepsilon > 0) \\
&\leq C \|\mathbf{m}\|_{H^k}^3
\end{aligned} \tag{34}$$

if  $k > N/2 + 1$ , where we used the fact  $\mathbf{u} = (1 - \alpha \Delta)^{-1} \mathbf{m}$ , and therefore  $\|\mathbf{u}\|_{H^s} \leq \|\mathbf{m}\|_{H^{s-2}}$  for all  $s \in \mathbb{R}$ .

$$\begin{aligned}
J_2 &= \frac{1}{2} \sum_{|\sigma| \leq k} \int_{\mathbb{R}^N} (\mathbf{u} \cdot \nabla) |D^\sigma \mathbf{m}|^2 \, dx = -\frac{1}{2} \sum_{|\sigma| \leq k} \int_{\mathbb{R}^N} (\operatorname{div} \mathbf{u}) |D^\sigma \mathbf{m}|^2 \, dx \\
&\leq C \|\nabla \mathbf{u}\|_{L^\infty} \|\mathbf{m}\|_{H^k}^2 \leq C \|\mathbf{m}\|_{H^{N/2-1+\varepsilon}} \|\mathbf{m}\|_{H^k}^2 \quad (\forall \varepsilon > 0) \\
&\leq C \|\mathbf{m}\|_{H^k}^3
\end{aligned} \tag{35}$$

if  $k > N/2 - 1$ . The estimates of  $I_2, I_3$  are simpler, and we have

$$\begin{aligned}
I_2 + I_3 &\leq \|(\nabla \mathbf{u})^\top \mathbf{m}\|_{H^k} \|\mathbf{m}\|_{H^k} \leq C(\|\nabla \mathbf{u}\|_{L^\infty} \|\mathbf{m}\|_{H^k} + \|\mathbf{u}\|_{H^{k+1}} \|\mathbf{m}\|_{L^\infty}) \|\mathbf{m}\|_{H^k} \\
&\leq C(\|\mathbf{m}\|_{H^{N/2-1+\varepsilon}} \|\mathbf{m}\|_{H^k} + \|\mathbf{m}\|_{H^{k-1}} \|\mathbf{m}\|_{H^{N/2+\varepsilon}}) \|\mathbf{m}\|_{H^k} \\
&\leq C \|\mathbf{m}\|_{H^k}^3
\end{aligned} \tag{36}$$

if  $k > N/2$ . Summarizing the above estimates, we obtain

$$\frac{d}{dt} \|\mathbf{m}\|_{H^k}^2 \leq C \|\mathbf{m}\|_{H^k}^3$$

for  $k > N/2 + 1$ , which implies

$$\|\mathbf{u}(t)\|_{H^k} \leq \frac{C\|\mathbf{u}_0\|_{H^k}}{1 - C\|\mathbf{u}_0\|_{H^k}t} \quad \forall t \in [0, T), \text{ where } T = \frac{1}{C\|\mathbf{u}_0\|_{H^k}},$$

where  $k > N/2 + 3$ .

We now prove uniqueness of solution in this class. Let  $(\mathbf{u}_1, \mathbf{m}_1), (\mathbf{u}_2, \mathbf{m}_2)$  two solution pairs corresponding to initial data  $(\mathbf{u}_{1,0}, \mathbf{m}_{1,0}), (\mathbf{u}_{2,0}, \mathbf{m}_{2,0})$ . We set  $\mathbf{u} = \mathbf{u}_1 - \mathbf{u}_2$ , and so on. Subtracting the equation for  $(\mathbf{u}_2, \mathbf{m}_2)$  from that of  $(\mathbf{u}_1, \mathbf{m}_1)$ , we find that

$$\partial_t \mathbf{m} + \operatorname{div}(\mathbf{u}_1 \otimes \mathbf{m}) + \operatorname{div}(\mathbf{u} \otimes \mathbf{m}_2) + (\nabla \mathbf{u}_1)^\top \mathbf{m} + (\nabla \mathbf{u})^\top \mathbf{m}_2 = 0. \quad (37)$$

Let  $p > N$ . Taking  $L^2(\mathbb{R}^N)$  product of (37) with  $\mathbf{m}|\mathbf{m}|^{p-2}$ , we obtain

$$\begin{aligned} \frac{1}{p} \frac{d}{dt} \|\mathbf{m}(t)\|_{L^p}^p &= - \left(1 - \frac{1}{p}\right) \int_{\mathbb{R}^N} (\operatorname{div} \mathbf{u}_1) |\mathbf{m}|^p dx - \int_{\mathbb{R}^N} (\operatorname{div} \mathbf{u}) \mathbf{m}_2 \cdot \mathbf{m} |\mathbf{m}|^{p-2} dx \\ &\quad - \int_{\mathbb{R}^N} (\mathbf{u} \cdot \nabla) \mathbf{m}_2 \cdot \mathbf{m} |\mathbf{m}|^{p-2} dx - \int_{\mathbb{R}^N} (\nabla \mathbf{u}_1)^\top \mathbf{m} \cdot \mathbf{m} |\mathbf{m}|^{p-2} dx \\ &\quad - \int_{\mathbb{R}^N} (\nabla \mathbf{u})^\top \mathbf{m}_2 \cdot \mathbf{m} |\mathbf{m}|^{p-2} dx \\ &\leq C(\|\operatorname{div} \mathbf{u}_1\|_{L^\infty} \|\mathbf{m}\|_{L^p}^p + \|\nabla \mathbf{u}\|_{L^\infty} \|\mathbf{m}_2\|_{L^p} \|\mathbf{m}\|_{L^p}^{p-1} + \|\mathbf{u}\|_{L^p} \|\nabla \mathbf{m}_2\|_{L^\infty} \|\mathbf{m}\|_{L^p}^{p-1} \\ &\quad + \|\nabla \mathbf{u}_1\|_{L^\infty} \|\mathbf{m}\|_{L^p}^p + \|\nabla \mathbf{u}\|_{L^\infty} \|\mathbf{m}_2\|_{L^p} \|\mathbf{m}\|_{L^p}^{p-1}) \\ &\leq C(\|\mathbf{u}_1\|_{H^k} + \|\mathbf{u}_2\|_{H^k}) \|\mathbf{m}\|_{L^p}^p \end{aligned}$$

for  $k > N/2 + 3$ . Hence,

$$\|\mathbf{m}(t)\|_{L^p} \leq \|\mathbf{m}_0\|_{L^p} \exp \left( C \int_0^t (\|\mathbf{u}_1(\tau)\|_{H^k} + \|\mathbf{u}_2(\tau)\|_{H^k}) d\tau \right).$$

This inequality implies the desired uniqueness of solutions in the class  $L^1(0, T; H^k(\mathbb{R}^N))$  with  $k > N/2 + 3$ . This gives (ii). The proof of (iii) was explained at the end of Section 2. This completes the proof of Theorem 1.  $\square$